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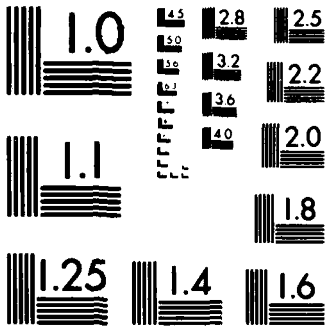

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

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More Limit Theory for the Sample Correlation
Function of Moving Averages

by

Richard Davis

and

Sidney Resnick

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Let $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ be a moving average process where $\{Z_t\}$ is iid with common distribution in the domain of attraction of a stable law with index α, $0<\alpha\leq 2$. If $0<\alpha<2$, $E|Z_1|^\alpha < \infty$ and the distribution of $|Z_1|$ and $|Z_1 Z_2|$ are tail equivalent then the sample correlation function of $\{X_t\}$ suitably normalized converges in distribution to the ratio of two dependent stable random variables with indices α and $\alpha/2$. This is in sharp contrast to the case $E|Z_1|^\alpha = \infty$ where the limit distribution is that of the ratio of two independent stable variables. Proofs rely heavily on point process techniques. The authors also consider the case when the sample correlations are asymptotically normal and extend slightly the classical result.

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More Limit Theory for the Sample Correlation Function of Moving Averages

By

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Abstract

Let $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ be a moving average process where $\{Z_t\}$ is iid with common distribution in the domain of attraction of a stable law with index α, $0 < \alpha \le 2$. If $0 < \alpha < 2$, $E|Z_1|^\alpha < \infty$ and the distribution of $|Z_1|$ and $|Z_1 Z_2|$ are tail equivalent then the sample correlation function of $\{X_t\}$ suitably normalized converges in distribution to the ratio of two dependent stable random variables with indices α and $\alpha/2$. This is in sharp contrast to the case $E|Z_1|^\alpha = \infty$ where the limit distribution is that of the ratio of two independent stable variables. Proofs rely heavily on point process techniques. We also consider the case when the sample correlations are asymptotically normal and extend slightly the classical result.

Short title: Limit Theory for Moving Averages

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More Limit Theory for the Sample Correlation

Function of Moving Averages

1. Introduction

Suppose $\{Z_t, -\infty < t < \infty\}$ is an independent, identically distributed (iid) sequence of random variables with regularly varying tail probabilities. More specifically assume

$$(1.1) \qquad P(|Z_1| > x) = x^{-\alpha}L(x)$$

with $\alpha > 0$ and L a slowly varying function at ∞ and also assume the tail balancing condition

$$(1.2) \qquad \frac{P(Z_1 > x)}{P(|Z_1| > x)} \to p \text{ and } \frac{P(Z_1 < -x)}{P(|Z_1| > x)} \to q$$

as $x \to \infty$, $0 \leq p \leq 1$ and $q = 1 - p$.

Given a sequence of real numbers $\{c_j, -\infty < j < \infty\}$ satisfying mild conditions (which for instance are always satisfied for ARMA processes) the moving averages

$$(1.3) \qquad \{X_t, -\infty < t < \infty\} = \{\sum_{j=-\infty}^{\infty} c_j Z_{t-j}, -\infty < t < \infty\}$$

exist as a strictly stationary sequence. The sample correlations of $\{X_t\}$ defined by

$$(1.4) \qquad \hat{\rho}(h) = \sum_{t=1}^{n-h} X_t X_{t+h} / \sum_{t=1}^{n} X_t^2, \; h > 0$$

are the objects of study in this paper.

In two previous papers (Davis and Resnick, 1984a, b) the weak limit behavior of functionals of $\{X_t\}$, including the sample covariance function, was discussed. In Davis and Resnick (1984b) the asymptotic limit distribution for $\hat{\rho}(h)$ was derived under the assumption that $E|Z_1|^{\alpha} = \infty$ and $0 < \alpha < 2$ and in

particular it was shown that there exists a slowly varying function $L_1(t)$ such that if $\rho(h) := \sum_{j=-\infty}^{\infty} c_j\, c_{j+h} / \sum_{j=-\infty}^{\infty} c_j^2$, then

$$n^{1/\alpha}\, L_1(n)(\hat{\rho}(h) - \rho(h))$$

converges in distribution to the ratio of two independent stable random variables with indices α and $\alpha/2$ respectively. Joint limit behavior of $\hat{\rho}(h)$ at various lags was also given. The asymptotic behavior of $\hat{\rho}(h)$ was found to depend on the weak limit behavior of the vector $(\sum_{t=1}^{n} Z_t\, Z_{t+\ell},\ \ell = 0, \ldots, h)$. This vector converged to a vector of independent, non-normal stable random variables.

In contrast to the above case when $E|Z_1|^\alpha = \infty$ we suppose in the present paper that $E|Z_1|^\alpha < \infty$ and in section 3 we obtain the surprising result that $\hat{\rho}(h)$ suitably normalized converges to a ratio of dependent stable random variables. Joint limit behavior of $(\hat{\rho}(\ell),\ 1 \leq \ell \leq h)$ is also given and as before depends on the behavior of $(\sum_{t=1}^{n} Z_t\, Z_{t+\ell},\ \ell = 0, \ldots, h)$. Again there is a clear distinction between the case where the α-moment is finite or infinite since in the case $E|Z_1|^\alpha < \infty$ we find this vector of sums of products converges weakly to a vector of dependent stable random variables. Both the results and the methods of proof are very different depending on whether $E|Z_1|^\alpha$ is finite or not.

Section 3 discusses these results which depend on point process methods. In addition we discuss some necessary results about tail behavior of products of random variables which is a class of problems which has received significant attention in analytic probability research. See Breiman (1965), Embrechts and Goldie (1980, 1982), Cline (1983). Also in section 2 we establish the asymptotic normality of $\hat{\rho}(h)$ under the assumption $E\, Z_1^2\, 1_{[|Z_1| \leq t]}$ is slowly varying at ∞. Our results and methods unify and extend classical results where a finite variance for Z_1 is assumed.

2. Sample correlations in the normal case

Let $\{Z_t\}$ be an iid sequence of random variables. Assume Z_1 belongs to the domain of attraction of a normal distribution which is equivalent to (cf. Feller, 1971, p. 313) the slow variation at infinity of the truncated second moment $L(t) = E\, Z_1^2\, 1_{[|Z_1|\le t]}$. This in turn is equivalent to

$$(2.1) \qquad \frac{t^2 P(|Z_1|>t)}{L(t)} \to 0 \text{ as } t \to \infty.$$

Moreover if the sequence of constants $\alpha_n \ge 0$ is chosen so that

$$(2.2) \qquad n\, \alpha_n^{-2} L(\alpha_n) \to 1,$$

then

$$\alpha_n^{-1} \sum_{t=1}^{n} (Z_t - E\, Z_t) \Rightarrow N(0,\, 1)$$

($\Rightarrow$ denotes convergence in distribution). If $\sigma^2 = \mathrm{Var}(Z_1) < \infty$ then we may choose $\alpha_n = n^{\frac{1}{2}}\sigma$ otherwise we have $\alpha_n = n^{\frac{1}{2}}\, \tilde{L}(n)$ where $\tilde{L}$ is slowly varying with $\tilde{L}(t) \to \infty$ as $t \to \infty$.

The goal of this section is to derive the limit distribution of the sample correlation function of the process $X_t = \sum_{j=-\infty}^{\infty} c_j\, Z_{t-j}$. As will be shown below, the weak limit of the sample correlation function can be determined via the limit distribution of the vector of partial sums $(\sum_{t=1}^{n} Z_t\, Z_{t+1}, \ldots, \sum_{t=1}^{n} Z_t\, Z_{t+h})$.

First we note that $Z_1\, Z_2$ also belongs to the domain of attraction of a normal distribution (Maller, 1981) which is obvious only if $\mathrm{Var}(Z_1) < \infty$. Now let $\beta_n > 0$ be chosen so that

$$(2.3) \qquad \beta_n^{-2} n\, E|Z_1\, Z_2|^2\, 1_{[|Z_1\, Z_2| \le \beta_n]} \to 1$$

and observe that if $\mathrm{Var}(Z_1) < \infty$ and $E\, Z_1 = 0$ then we may take $\beta_n = n^{\frac{1}{2}}\sigma^2$. On the other hand if $\mathrm{Var}(Z_1) = \infty$, then (Maller, 1981)

$$(2.4) \qquad \alpha_n/\beta_n \to 0.$$

Proposition 2.1. Let $\{Z_t\}$ be iid with zero mean and assume $L(t) = E\, Z_1^2\, 1_{[|Z_1| \leq t]}$ is slowly varying. Then for any fixed positive integer h

$$(\beta_n^{-1} \sum_{t=1}^{n} Z_t\, Z_{t+\ell},\ 1 \leq \ell \leq h) \Rightarrow (N_1, N_2, \ldots, N_h)$$

where $N_1, N_2, \ldots, N_h$ are iid $N(0, 1)$ random variables.

Proof. First we show that the kh dimensional vector

$$(Z_t\, Z_{t+\ell},\ 1 \leq \ell \leq h,\ 1 \leq t \leq k)$$

belongs to the domain of attraction of a multivariate normal distribution with independent components. By Theorem 3.2 in de Haan, Omey, and Resnick (1984) it suffices to show for $s \neq t$ or $i \neq \ell$

$$n\beta_n^{-2}\, E(Z_t\, Z_{t+\ell}\, 1_{[|Z_t\, Z_{t+\ell}| \leq \beta_n x]}\, Z_s\, Z_{s+i}\, 1_{[|Z_s\, Z_{s+i}| \leq \beta_n y]}) \tag{2.5}$$
$$\to 0 \text{ as } n \to \infty \text{ for all } x > 0 \text{ and } y > 0.$$

If $Z_t\, Z_{t+\ell}$ and $Z_s\, Z_{s+i}$ are independent, then (2.5) is automatic by (2.3) and the assumption $E\, Z_t = 0$. On the other hand if $Z_t\, Z_{t+\ell}$ and $Z_s\, Z_{s+i}$ are not independent, then (2.5) is equal to

$$n\beta_n^{-2}\, E(Z_1\, Z_2\, 1_{[|Z_1\, Z_2| \leq \beta_n x]}\, Z_2\, Z_3\, 1_{[|Z_2\, Z_3| \leq \beta_n y]})$$
$$= n\beta_n^{-2}\, E(Z_1\, Z_2\, 1_{[|Z_1\, Z_2| \leq \beta_n x]}\, Z_2\, Z_3\, 1_{[|Z_2\, Z_3| \leq \beta_n y]}\, 1_{[|Z_2| > \alpha_n]})$$
$$+ n\beta_n^{-2}\, E(Z_1\, Z_2\, 1_{[|Z_1\, Z_2| \leq \beta_n x]}\, Z_2\, Z_3\, 1_{[|Z_2\, Z_3| \leq \beta_n y]} 1_{[|Z_2| \leq \alpha_n]})$$
$$= A + B.$$

We have

$$|A| \leq n\beta_n^{-2}\, \beta_n^2 (x \vee y) P(|Z_2| > \alpha_n)$$
$$= (x \vee y) nP(|Z_2| > \alpha_n) \to 0$$

by (2.1) and (2.2). As for B, we have

$$|B| = n\beta_n^{-2} \int_0^{\alpha_n} t^2 (E\, Z_1\, 1_{[|Z_1| \leq \beta_n x/t]}\, E\, Z_3\, 1_{[|Z_3| \leq \beta_n y/t]}) P(|Z_2| \epsilon dt).$$

If $\text{Var}(Z_t) < \infty$, then by the dominated convergence theorem, (note $n\beta_n^{-2} \to \sigma^{-4}$), $|B| \to 0$. If $\text{Var}(Z_t) = \infty$, then

$$
\begin{aligned}
|B| &\leq (E|Z_1|)^2 \; n\beta_n^{-2} \; E\, Z_1^2 \, 1_{[|Z_1| \leq \alpha_n]} \\
&= (E|Z_1|)^2 (\alpha_n/\beta_n)^2 \; n\alpha_n^{-2} \; E\, Z_1^2 \, 1_{[|Z_1| \leq \alpha_n]} \\
&\to 0
\end{aligned}
$$

by (2.2) and (2.4), which establishes (2.5).

Now for a fixed $\underset{\sim}{\lambda} \in \mathbb{R}^h$, define the h-dependent sequence by

$$Y_t = \lambda_1 Z_t Z_{t+1} + \lambda_2 Z_t Z_{t+2} + \ldots + \lambda_h Z_t Z_{t+h}.$$

For each fixed integer $k > 2h$, we have

$$\sum_{t=1}^{n} Y_t = \sum_{i=1}^{r} U_i + \sum_{i=1}^{r-1} V_i + (Y_{rk-h+1} + \ldots + Y_n)$$

where $U_i = (Y_{(i-1)k+1} + \ldots + Y_{ik-h})$, $V_i = (Y_{ik-h+1} + \ldots + Y_{ik})$ and $r = [n/k]$. The U_i are iid and by applying the first part of the proof, we have

$$\beta_r^{-1} \sum_{i=1}^{r} U_i \Rightarrow N(0,\ (k-h)(\lambda_1^2 + \ldots + \lambda_h^2))$$

and since $\beta_r/\beta_n \to k^{-\frac{1}{2}}$

$$\beta_n^{-1} \sum_{i=1}^{r} U_i \Rightarrow N(0,\ (1-h/k)(\lambda_1^2 + \ldots + \lambda_h^2)).$$

The same reasoning also gives

$$\beta_n^{-1} \sum_{i=1}^{r} V_i \Rightarrow N(0, (h/k)(\lambda_1^2 + \ldots + \lambda_h^2)).$$

The piece $(Y_{rk-h} + \ldots + Y_n)$ is a sum of at most 2h terms and hence is $o_p(\beta_n)$ so that for every $\varepsilon > 0$

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\left(\beta_n^{-1} \left| \sum_{t=1}^{n} Y_t - \sum_{i=1}^{r} U_i \right| > \varepsilon\right) = 0.$$

Finally, we have $N(0, (1-h/k)(\lambda_1^2 + \dots + \lambda_h^2)) \Rightarrow N(0, \lambda_1^2 + \dots + \lambda_h^2)$ as $k \to \infty$ so that by Theorem 4.2 in Billingsley (1968),

$$\beta_n^{-1} \sum_{t=1}^{n} Y_t = \sum_{i=1}^{h} \lambda_i \, \beta_n^{-1} \sum_{t=1}^{n} Z_t \, Z_{t+i} \Rightarrow N(0, \lambda_1^2 + \dots + \lambda_h^2)$$

which completes the proof by an appeal to the Cramér-Wold device. $\square$

We now consider the moving average process

$$X_t = \sum_{j=-\infty}^{\infty} c_j \, Z_{t-j}$$

where the Z_t's satisfy the assumptions of Proposition 2.1. The coefficients are assumed to satisfy

$$(2.6) \qquad \sum_{j=-\infty}^{\infty} |c_j| < \infty \text{ and } \sum_{j=-\infty}^{\infty} |c_j|^{\gamma} |j| < \infty$$

where $\gamma = 2$ if $\mathrm{Var}(Z_1) < \infty$, otherwise $0 < \gamma < 2$. Define for $h \geq 0$ the sample correlation function

$$\hat{\rho}(h) = C(h)/C(0)$$

where $C(h) = \sum_{t=1}^{n} X_t \, X_{t+h}$. Set $\rho(h) = \sum_{j=-\infty}^{\infty} c_j \, c_{j+h} / \sum_{j=-\infty}^{\infty} c_j^2$ which is the correlation function for $\{X_t\}$ if $\mathrm{Var}(Z_1) < \infty$.

We first show that

$$(2.7) \qquad \alpha_n^{-2} \, C(0) \xrightarrow{P} \sum_{i=-\infty}^{\infty} c_i^2 .$$

This will be accomplished by establishing

$$(2.8) \qquad \alpha_n^{-2} \left(C(0) - \sum_{t=1}^{n} \sum_{i=-\infty}^{\infty} c_i^2 \, Z_{t-i}^2\right) \xrightarrow{P} 0,$$

$$(2.9) \qquad \alpha_n^{-2} \left(\sum_{t=1}^{n} \sum_{i=-\infty}^{\infty} c_i^2 \, Z_{t-i}^2 - \sum_{i=-\infty}^{\infty} c_i^2 \sum_{t=1}^{n} Z_t^2\right) \xrightarrow{P} 0,$$

and

$$(2.10) \qquad \alpha_n^{-2} \sum_{t=1}^{n} Z_t^2 \xrightarrow{P} 1.$$

The difference in (2.8) is $\alpha_n^{-2} \sum_{t=1}^{n} \sum_{\substack{i,j \\ i \neq j}} c_i \, c_j \, Z_{t-i} \, Z_{t-j}$ and if $\mathrm{Var}(Z_1) = \infty$ then $n/\alpha_n^2 \to 0$ so that $E|\alpha_n^{-2} \sum_{t=1}^{n} \sum_{\substack{i,j \\ i \neq j}} c_i \, c_j \, Z_{t-i} \, Z_{t-j}| \le n/\alpha_n^2 (\sum_{i=-\infty}^{\infty} |c_i|)^2 E|Z_1 \, Z_2| \to 0$ proving (2.8). If $\mathrm{Var}(Z_1) < \infty$ then it is easy to show that the variance of the difference in (2.8) goes to zero. The difference in (2.9) is $\alpha_n^{-2} \sum_{i=-\infty}^{\infty} c_i^2 \, U_{n,i}$ where $U_{n,i} = \sum_{t=1-i}^{n-i} Z_t^2 - \sum_{t=1}^{n} Z_t^2$ is a sum of at most $2i$ random variables. Since $E|Z_1|^\gamma < \infty$ where γ is defined in (2.6), we have

$$E|\alpha_n^{-2} \sum_{i=-\infty}^{\infty} c_i^2 \, U_{n,i}|^{\gamma/2} \le 2\alpha_n^{-\gamma} \sum_{i=-\infty}^{\infty} |c_i|^\gamma |i| \; E|Z_1|^\gamma \to 0$$

giving (2.9). Finally the weak law of large numbers yields (2.10) (cf. Theorem 2, p. 236, Feller, 1971).

For a fixed $\ell \ge 1$, set $\psi_{i,j} = c_i(c_{i-j+\ell} - c_{i-j}\rho(\ell)), i = 0, \pm 1, \pm 2, \ldots,$ $j = \pm 1, \pm 2, \ldots$. Then

$$(2.11) \qquad \alpha_n^2 \beta_n^{-1}(\hat{\rho}(\ell) - \rho(\ell) - [C(0)]^{-1} \sum_{t=1}^{n} \sum_{j \neq 0} \sum_{i=-\infty}^{\infty} \psi_{i,j} \, Z_{t-i} \, Z_{t-i+j}) \xrightarrow{P} 0$$

and for each $j > 0$

$$(2.13) \qquad \begin{aligned} &\alpha_n^2 \beta_n^{-1}(\sum_{t=1}^{n} (\sum_{i=-\infty}^{\infty} \psi_{i,j} \, Z_{t-i} \, Z_{t-i+j} + \sum_{i=-\infty}^{\infty} \psi_{i,-j} Z_{t-i} \, Z_{t-i-j}) \\ &- \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j}) \sum_{t=1}^{n} Z_t \, Z_{t+j}) \xrightarrow{P} 0. \end{aligned}$$

These two results together with Proposition 2.1 and the continuous mapping theorem suggest that

$$\alpha_n^2\beta_n^{-1}(\hat{\rho}(\ell) - \rho(\ell) \Rightarrow \sum_{j=1}^{\infty}(\sum_{i=-\infty}^{\infty}(\psi_{i,j} + \psi_{i,-j})N_j / \sum_{i=-\infty}^{\infty} c_i^2$$

$$= \sum_{j=1}^{\infty}(\rho(\ell+j) + \rho(\ell-j) - 2\rho(j)\rho(\ell))N_j$$

where N_1, N_2, ... are iid N(0, 1) random variables. This is in fact the content of the following theorem. The proof of this theorem as well as (2.11) and (2.12) are only slight modifications (take $\delta = \gamma$) of the arguments given for Proposition 4.1, Proposition 4.3(i) and Theorem 4.4 of Davis and Resnick (1984b) and hence are omitted.

Theorem 2.2. Suppose $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ where $\{c_j\}$ satisfies (2.6) and $\{Z_t\}$ is an iid sequence with zero mean and $L(t) = E\, Z_1^2\, 1_{[|Z_1| \le t]}$ is slowly varying. If α_n and β_n are chosen to satisfy (2.2) and (2.3) respectively, then for any $h \ge 1$

$$\alpha_n^2\beta_n^{-1}(\hat{\rho}(\ell) - \rho(\ell),\ 1 \le \ell \le h) \Rightarrow (\sum_{j=1}^{\infty}(\rho(\ell+j) + \rho(\ell-j) - 2\rho(j)\rho(\ell))N_j,\ 1 \le \ell \le h)$$

where N_1, N_2, ... are iid N(0, 1).

3. Sample correlations and regularly varying tail probabilities

In this section we examine the weak limit behavior of $(\sum_{t=1}^{n} Z_t Z_{t+\ell},\ 0 \leq \ell \leq h)$ under the assumptions that $\{Z_t\}$ is iid satisfying (1.1), (1.2) with $0 < \alpha < 2$ and also that $E|Z_1|^\alpha < \infty$ and

$$(3.1) \qquad \lim_{t\to\infty} \frac{P(|Z_1 Z_2| > t)}{P(|Z_1| > t)} = 2E|Z_1|^\alpha.$$

Thus the distribution of $\log|Z_1|$ is in the class S_α studied by several authors (Chover, Ney, Wainger, 1973; Embrechts and Goldie, 1980, 1982). If a finite limit in (3.1) exists, it must be of the form $2E|Z_1|^\alpha$ (Chover, et al, 1973). In studying the behavior of $\sum_{t=1}^{n} Z_t Z_{t+\ell}$ it becomes clear a limit distibution does not exist when $E|Z_1|^\alpha < \infty$ without (3.1). It is interesting that the class S_α achieves interest from an additional perspective. See also Cline (1983).

It is difficult to get a decent characterization of when (3.1) holds but the following needed fact can be gleaned quickly. (Cf. Embrechts and Goldie, 1982 and Cline, 1983.) Suppose (1.1) holds and we write for $t > s > 0$

$$\begin{aligned} P(|Z_1 Z_2| > t) &= P(|Z_1 Z_2| > t,\ |Z_2| \leq s) \\ &\quad + P(|Z_1 Z_2| > t,\ s < |Z_2| \leq t/s) \\ &\quad + P(|Z_1 Z_2| > t,\ |Z_2| > t/s) = I + II + III. \end{aligned}$$

Now III may be written as

$$\begin{aligned} &P(|Z_1 Z_2| \wedge |Z_2 s| > t) \\ &\quad = P\big((|Z_1| \wedge |s|)|Z_2| > t\big). \end{aligned}$$

Since $|Z_1| \wedge s$ is bounded, a result of Breiman (1965) (a simple dominated convergence argument; see also Cline, 1983) gives the above asymptotic to

$$\sim E(|Z_1 \wedge s|^\alpha)P(|Z_2| > t)$$

as $t \to \infty$. For I we have

$$I/P(|Z_1| > t) = \int_0^s \frac{P(|Z_1| > t/y)}{P(|Z_1| > t)} P(|Z_2| \in dy)$$

and so by regular variation and dominated convergence we get as $t \to \infty$

$$I \sim \int_0^s y^\alpha P(|Z_2| \in dy)P(|Z_1| > t).$$

(By letting $s \to \infty$, we see that $\liminf_{t\to\infty} P(|Z_1 Z_2| > t)/P(|Z_1| > t) \geq 2E|Z_1|^\alpha$.)

Since we know by the Chover, Ney, Wainger (1973) result that the only possible finite limit in (3.1) is $2E(|Z_1|^\alpha)$ we obtain the following result.

Proposition 3.1. Suppose (1.1) holds and $E(|Z_1|^\alpha) < \infty$. Then (3.1) holds iff

(3.2) $$\lim_{s\to\infty} \limsup_{t\to\infty} P(|Z_1 Z_2| > t, s < |Z_1| \leq t/s)/P(|Z_1| > t) =$$

$$\lim_{s\to\infty} \limsup_{t\to\infty} \int_s^{t/s} \frac{P(|Z_1| > t/y)}{P(|Z_1| > t)} P(|Z_1| \in dy) = 0.$$

We now commence a study of the weak limit behavior of $(\sum_{t=1}^n Z_t Z_{t+\ell}, 0 \leq \ell \leq h)$.

The method of attack uses point processes and useful background on this subject and its relevance to limit theory is contained in Neveu (1976), Kallenberg (1976), Resnick (1984), Davis and Resnick (1984a, b). We set $M_p(E)$ equal to the space of point measures on the Euclidean space E and metrize $M_p(E)$ by the vague metric. A point measure on E is a Radon measure on E of the form $\sum_i \epsilon_{x_i}$ where $x_i \in E$ and for a Borel subset $B \subset E$ we have $\epsilon_x(B) = 1$ if $x \in B$; 0 otherwise. A Poisson process on E with mean or intensity measure μ will be denoted PRM(μ); i.e. Poisson random measure with mean measure μ.

We begin by supposing $\overline{R}_0 := [-\infty, 0) \cup (0, \infty]$ is topologized so that neighborhoods of $\pm\infty$ are compact sets; i.e. on the positive half axis the usual roles of 0

and ∞ are interchanged and similarly for the negative half axis. Set $E = \overline{\mathbb{R}}_o \times \mathbb{R}^{2h}$ and suppose $\sum_{k=1}^{\infty} \epsilon_{j_k}$ is PRM on $\overline{\mathbb{R}}_o$ with mean measure $\lambda(dx) = (p\alpha x^{-\alpha-1} 1_{(0,\infty]}(x) + q\alpha(-x)^{-\alpha-1} 1_{(-\infty,0]}(x))dx$. Lastly define a_n to satisfy

(3.3) $\quad a_n = \inf\{x: P(|Z_1| > x) \le n^{-1}\}.$

Proposition 3.2. Suppose (1.1) and (1.2) hold and set

$$\mathcal{Z}_{k,n} = (a_n^{-1} Z_k, Z_{k-h}, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{k+h})$$

and suppose further that

$$\{\mathcal{Z}^{(k)}\} = \{(Z_{-h}^{(k)}, \ldots, Z_{-1}^{(k)}, Z_1^{(k)}, \ldots, Z_h^{(k)})\}$$

is an iid sequence of $\mathbb{R}^{2h}$ valued random elements and the components in $\mathcal{Z}^{(k)}$ are iid and distributed as Z_1. The sequence $\{\mathcal{Z}^{(k)}\}$ is assumed independent of the point process $\sum_{k=1}^{\infty} \epsilon_{j_k}$. Then the following weak convergence result in $M_p(E)$ holds as $n \to \infty$

$$\sum_{k=1}^{n} \epsilon_{\mathcal{Z}_{k,n}} \Rightarrow \sum_{k=1}^{\infty} \epsilon_{(j_k, \mathcal{Z}^{(k)})}.$$

Remark. The limit point process is PRM on E with mean measure $\mu(dx, dx_{-h}, dx_{-h+1}, \ldots, dx_{-1}, dx_1, \ldots, dx_h) := \lambda(dx)F(dx_{-h})F(dx_{-h+1})\ldots F(dx_{-1})F(dx_1)\ldots F(dx_h)$.

Proof. It suffices by Theorem 4.7 in Kallenberg (1976) to show as $n \to \infty$

(3.4) $\quad nP(\mathcal{Z}_{1,n} \in A \times B^{2h}) \to \mu(A \times B^{2h})$

where $A = (x, \infty]$ or $(-\infty, -x]$, $x > 0$ and B^{2h} is a bounded 2h dimensional rectangle and also

(3.5) $\quad P(\sum_{k=1}^{n} \epsilon_{\mathcal{Z}_{k,n}} (R) = 0) \to P(\sum_{k=1}^{\infty} \epsilon_{(j_k, \mathcal{Z}^k)} (R) = 0)$

where R is a finite union of bounded rectangles in E. Since in case $A = (x, \infty]$ $nP(\mathcal{Z}_{1,n} \in A \times B^{2h}) = nP(a_n^{-1} Z_1 > x)F^{\otimes 2h}(B^{2h})$ ($F^{\otimes 2h}$ is product measure), we obtain

(3.4) immediately from (1.1), (1.2) and (3.3) so we focus on (3.5). Since the sequence $\{Z_{i,n}, -\infty < i < \infty\}$ is $2h + 1$ dependent a standard argument (cf. Leadbetter, Lindgren and Rootzen, 1983, chapters 3, 5) yields

$$(3.6) \qquad P^k\Big(\sum_{i=1}^{[n/k]} \epsilon_{\mathcal{Z}_{i,n}}(R) = 0\Big) - P\Big(\sum_{i=1}^{n} \epsilon_{\mathcal{Z}_{i,n}}(R) = 0\Big) \to 0,$$

for any k as $n \to \infty$. Furthermore by a Bonferroni inequality

$$[n/k]P(\mathcal{Z}_{1,n} \in R) - [n/k]\sum_{i=2}^{[n/k]} P(\mathcal{Z}_{1,n} \in R, \mathcal{Z}_{i,n} \in R)$$

$$\leq P\Big(\bigcup_{i=1}^{[n/k]} [Z_{i,n} \in R]\Big) = P\Big(\sum_{i=1}^{[n/k]} \epsilon_{\mathcal{Z}_{i,n}}(R) > 0\Big)$$

$$\leq [n/k]P(\mathcal{Z}_{1,n} \in R).$$

Since $P(\mathcal{Z}_{1,n} \in R, \mathcal{Z}_{i,n} \in R)$ can be dominated by a probability based only on $a_n^{-1} Z_1$ and $a_n^{-1} Z_i$ it readily follows that

$$\limsup_{n\to\infty} [n/k] \sum_{i=2}^{[n/k]} P(\mathcal{Z}_{1,n} \in R, \mathcal{Z}_{i,n} \in R) = o(1/k)$$

and hence applying the natural generalization of (3.4) to finite unions of disjoint rectangles we obtain

$$1 - k^{-1}\mu(R) \leq \liminf_{n\to\infty} P\Big(\sum_{i=1}^{[n/k]} \epsilon_{\mathcal{Z}_{i,n}}(R) = 0\Big)$$

$$\leq \limsup_{n\to\infty} P\Big(\sum_{i=1}^{[n/k]} \epsilon_{\mathcal{Z}_{i,n}}(R) = 0\Big)$$

$$\leq (1 - k^{-1}\mu(R) + o(1/k))$$

for any k. Raising all sides of this inequality to the kth power, letting $k \to \infty$ and then applying (3.4) gives the desired conclusion (3.5) since we obtain from (3.6)

$$P\Big(\sum_{i=1}^{n} \epsilon_{\mathcal{Z}_{i,n}}(R) = 0\Big) \to e^{-\mu(R)} = P\Big(\sum_{k=1}^{\infty} \epsilon_{(j_k, \mathcal{Z}_k)}(R) = 0\Big). \qquad \square$$

Corollary 3.3. If (1.1) and (1.2) hold then

$$\left(\sum_{k=1}^{n} \epsilon_{a_n^{-2} Z_k^2}, \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k-\ell})}, \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k+\ell})}, \ \ell = 1, \ldots, h\right)$$

$$\Rightarrow \left(\sum_{k=1}^{\infty} \epsilon_{j_k^2}, \sum_{k=1}^{\infty} \epsilon_{(j_k, Z_{-\ell}^{(k)})}, \sum_{k=1}^{\infty} \epsilon_{(j_k, Z_{\ell}^{(k)})}, \ \ell = 1, \ldots, h\right)$$

in $M_p(\bar{\mathbb{R}}_o) \times (M_p(\bar{\mathbb{R}}_o \times \mathbb{R}))^{2h}$ where the last factor is the 2h-fold Cartesian product of $M_p(\bar{\mathbb{R}}_o \times \mathbb{R})$.

Proof. First restrict the state space in Proposition 3.2 to the compact set $K_s = \{z: |z| > s^{-1}\} \times [-s, s]^{2h}$ to obtain

$$\sum_{k=1}^{n} \epsilon_{\underset{\sim}{Z}_{k,n}}(K_s \cap \cdot) \Rightarrow \sum_{k=1}^{\infty} \epsilon_{(j_k, \underset{\sim}{Z}^{(k)})}(K_s \cap \cdot).$$

Because the state space is compact we get by a variant of the continuous mapping theorem (Resnick, 1984, Proposition 1.1) that

$$\left(\sum_{k=1}^{n} \epsilon_{a_n^{-2} Z_k^2}, \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k-\ell})} 1_{[\underset{\sim}{Z}_{k,n} \in K_s]}, \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k+\ell})} 1_{[\underset{\sim}{Z}_{k,n} \in K_s]}, \ \ell = 1, \ldots, h\right)$$

$$\Rightarrow \left(\sum_{k=1}^{\infty} \epsilon_{j_k^2}, \sum_{k=1}^{\infty} \epsilon_{(j_k, Z_{-\ell}^{(k)})} 1_{[(j_k, \underset{\sim}{Z}^{(k)}) \in K_s]}, \sum_{k=1}^{\infty} \epsilon_{(j_k, Z_{\ell}^{(k)})} 1_{[(j_k, \underset{\sim}{Z}^{(k)}) \in K_s]}, \ \ell = 1, \ldots, h\right).$$

Since the right side above converges to the desired limit as $s \to \infty$ an application of Billingsley, 1968, Theorem 4.2 yields the result provided we show for any $\eta > 0$

$$\lim_{s\to\infty} \limsup_{n\to\infty} P\left[\rho\left(\sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k+u})}, \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k+u})} 1_{[\underset{\sim}{Z}_{k,n} \in K_s]}\right) > \eta\right] = 0$$

for any $u \in \{-h, \ldots, -1, 1, \ldots, h\}$ where ρ is the vague metric on $M_p(\bar{\mathbb{R}}_o \times \mathbb{R})$. Let f be non-negative and continuous with compact support on $\bar{\mathbb{R}}_o \times \mathbb{R}$; we take the support contained in $\{z: |z| > a^{-1}\} \times [-a, a]$ for some $a > 0$. Because of the definition of ρ (Kallenberg, 1976, page 95) it suffices to check

$$\lim_{s\to\infty} \limsup_{n\to\infty} P[|\sum_{k=1}^{n} f(a_n^{-1} Z_k, Z_{k+u}) - \sum_{k=1}^{n} f(a_n^{-1} Z_k, Z_{k+u}) 1_{[\mathcal{Z}_{k,n} \in K_s]}| > \eta]$$

$$= \lim_{s\to\infty} \limsup_{n\to\infty} P[\sum_{k=1}^{n} f(a_n^{-1} Z_k, Z_{k+u}) 1_{[\mathcal{Z}_{k,n} \in K_s]^c} > \eta] = 0.$$

The above probability is bounded by

$$P(\bigcup_{k=1}^{n} [|a_n^{-1} Z_k| > a^{-1}, |Z_{k+u}| \le a, \mathcal{Z}_{k,n} \notin K_s])$$

which by subadditivity and stationarity is

$$\le nP[|a_n^{-1} Z_o| > a^{-1}, |Z_u| \le a, \mathcal{Z}_{o,n} \notin K_s]$$

$$\le nP[|a_n^{-1} Z_o| > a^{-1}, |Z_u| \le a, |a_n^{-1} Z_o| \le s^{-1}]$$

$$+ \sum_{|\ell|=1}^{n} nP[|a_n^{-1} Z_o| > a^{-1}, |Z_u| \le a, |Z_\ell| > s].$$

The first term is zero if $s > a$ and the second piece with the summation is

$$\le 2h\, P[|a_n^{-1} Z_o| > a^{-1}]P[|Z_o| > s]$$

$$\to 2ha^{\alpha}P[|Z_o| > s]$$

as $n \to \infty$. Since this last term goes to zero as $s \to \infty$ the desired result is obtained. □

We now study a collection of point processes of products.

<u>Proposition 3.4</u>. Suppose (1.1), (1.2), (3.1) hold and $E|Z_1|^\alpha < \infty$. Then

$$(3.7) \qquad (\sum_{k=1}^{n} \epsilon_{a_n^{-2} Z_k^2}, \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k Z_{k+\ell})}, \ell = 1, \ldots, h) \Rightarrow$$

$$(\sum_{k=1}^{\infty} \epsilon_{j_k^2}, \sum_{k=1}^{\infty} (\epsilon_{j_k Z_{-\ell}^{(k)}} + \epsilon_{j_k Z_\ell^{(k)}}), \ell = 1, \ldots h)$$

in $M_p(\bar{\mathbb{R}}_o) \times M_p(\bar{\mathbb{R}}_o \times \mathbb{R})^h$.

Remark. For each $\ell \in (-h, \ldots, -1, 1, \ldots, h)$ we have $\sum_{k=1}^{\infty} \epsilon_{j_k Z_\ell^{(k)}}$ is PRM(ν) on $\bar{\mathbb{R}}_0$ and for $x > 0$

$$\nu(x, \infty) = x^{-\alpha}(pEZ_1^{\alpha} 1_{[Z_1 > 0]} + qE|Z_1|^{\alpha} 1_{[Z_1 < 0]}),$$

$$\nu(-\infty, -x) = |x|^{-\alpha}(qEZ_1^{\alpha} 1_{[Z_1 > 0]} + pE|Z_1|^{\alpha} 1_{[Z_1 < 0]}),$$

$$\nu\{z: |z| > x\} = x^{-\alpha}E|Z_1|^{\alpha}.$$

Proof. Making restrictions to the compact sets $K_{s,\gamma} = \{z: |z| > s^{-1}\} \times [-\gamma, \gamma]$ we get from Corollary 3.3

$$\Big(\sum_{k=1}^{n} \epsilon_{a_n^{-2} Z_k^2}, \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k-\ell})}(K_{s,\gamma} \cap \cdot), \sum_{k=1}^{n} \epsilon_{(a_n^{-1} Z_k, Z_{k+\ell})}(K_{\gamma,s} \cap \cdot), \quad \ell = 1, \ldots, h\Big)$$

$$\Rightarrow \Big(\sum_{k=1}^{\infty} \epsilon_{j_k^2}, \sum_{k=1}^{\infty} \epsilon_{(j_k, Z_{-\ell}^{(k)})}(K_{s,\gamma} \cap \cdot), \sum_{k=1}^{\infty} \epsilon_{(j_k, Z_\ell^{(k)})}(K_{\gamma,s} \cap \cdot), \quad \ell = 1, \ldots, h\Big).$$

The map $(x, y) \to xy$ applied to the points of a point measure with compact state space induces a continuous map on the space of point measures with common compact state space (Resnick, 1984, Proposition 1.1) and hence

$$(3.8) \qquad \Big(\sum_{k=1}^{n} \epsilon_{a_n^{-2} Z_k^2}, \sum_{k=1}^{n} \epsilon_{a_n^{-1} Z_k Z_{k-\ell}} 1_{[(a_n^{-1} Z_k, Z_{k-\ell}) \in K_{s,\gamma}]},$$

$$\sum_{k=1}^{n} \epsilon_{a_n^{-1} Z_k Z_{k+\ell}} 1_{[(a_n^{-1} Z_k, Z_{k+\ell}) \in K_{\gamma,s}]}, \quad \ell = 1, \ldots, h\Big)$$

$$\Rightarrow \Big(\sum_{k=1}^{\infty} \epsilon_{j_k^2}, \sum_{k=1}^{\infty} \epsilon_{j_k Z_{-\ell}^{(k)}} 1_{[(j_k, Z_{-\ell}^{(k)}) \in K_{s,\gamma}]},$$

$$\sum_{k=1}^{\infty} \epsilon_{j_k Z_\ell^{(k)}} 1_{[(j_k, Z_\ell^{(k)}) \in K_{\gamma,s}]}, \quad \ell = 1, \ldots, h\Big)$$

in $M_p((0, \infty]) \times M_p(\bar{\mathbb{R}}_0)^{2h}$. We now remove γ from the relation (3.8) by first noting that as $\gamma \to \infty$ the right side converges weakly to the corresponding expression with $\gamma = \infty$ and in order to apply Theorem 4.2 in Billingsley (1968) we must check

for f non-negative and continuous with compact support on $\bar{\mathbb{R}}_0$ and for $\ell = 1, \ldots, h$

$$(3.9) \qquad \lim_{\gamma\to\infty} \limsup_{n\to\infty} P\Big[\sum_{k=1}^{n} f(a_n^{-1} Z_k Z_{k-\ell}) 1_{[|a_n^{-1} Z_k| > s^{-1},\ |Z_{k-\ell}| > \gamma]} > \eta\Big] = 0$$

and

$$(3.10) \qquad \lim_{\gamma\to\infty} \limsup_{n\to\infty} P\Big[\sum_{k=1}^{n} f(a_n^{-1} Z_k Z_{k+\ell}) 1_{[|a_n^{-1} Z_k| \le \gamma^{-1},\ |Z_{k+\ell}| \le s]} > \eta\Big] = 0$$

for any $\eta > 0$. Supposing the support of f is $\{z: |z| > a\}$ we have the probability in (3.9) is bounded by

$$nP[|a_n^{-1} Z_k Z_{k-\ell}| > a,\ |a_n^{-1} Z_k| > s^{-1},\ |Z_{k-\ell}| > \gamma]$$

$$\le nP[|a_n^{-1} Z_k| > s^{-1}]P[|Z_{k-\ell}| > \gamma]$$

$$\to s^{\alpha}P[|Z_1| > \gamma]$$

which goes to zero as $\gamma \to \infty$. The probability in (3.10) is bounded by

$$nP[|a_n^{-1} Z_k Z_{k+\ell}| > a,\ |a_n^{-1} Z_k| \le \gamma^{-1}, |Z_{k+\ell}| \le s]$$

and for $s/\gamma < a$ this probability is zero. We may thus rewrite (3.8) with $\gamma = \infty$ as follows:

$$(3.11) \qquad \Big(\sum_{k=1}^{n} \epsilon_{a_n^{-1} Z_k^2},\ \sum_{k=1}^{n} \epsilon_{a_n^{-1} Z_k Z_{k-\ell}} 1_{[|a_n^{-1} Z_k| > s^{-1}]},\ \sum_{k=1}^{n} \epsilon_{a_n^{-1} Z_k Z_{k+\ell}} 1_{[|Z_{k+\ell}| \le s]},\ \ell = 1, \ldots, h\Big)$$

$$\Rightarrow \Big(\sum_{k=1}^{\infty} \epsilon_{j_k^2},\ \sum_{k=1}^{\infty} \epsilon_{j_k Z_{-\ell}^{(k)}} 1_{[|j_k| > s^{-1}]},\ \sum_{k=1}^{\infty} \epsilon_{j_k Z_{\ell}^{(k)}} 1_{[|Z_{\ell}^{(k)}| \le s]},\ \ell = 1, \ldots, h\Big).$$

On the left side of (3.11) take the process corresponding to $-\ell$ as subscript, change variables to $k' = k - \ell$, and add the result to the process corresponding to $+\ell$. After adjusting for $o_p(1)$ terms we get from (3.11)

$$(3.12)\qquad \Big(\sum_{k=1}^{n} \epsilon_{a_n^{-2} Z_k^2},\ \sum_{k=1}^{n} \epsilon_{a_n^{-1} Z_k Z_{k+\ell}} (1_{[|Z_{k+\ell}| \le s]} + 1_{[|a_n^{-1} Z_{k+\ell}| > s^{-1}]}) \qquad \ell = 1, \dots, h\Big)$$

$$\Rightarrow \Big(\sum_{k=1}^{n} \epsilon_{j_k^2},\ \sum_{k=1}^{\infty} (\epsilon_{j_k Z_\ell^{(k)}} 1_{[|Z_\ell^{(k)}| \le s]} + \epsilon_{j_k Z_{-\ell}^{(k)}} 1_{[|j_k| > s^{-1}]}),\qquad \ell = 1, \dots, h\Big).$$

As $s \to \infty$ the right side of (3.12) converges weakly to the right side of (3.7). The desired result (3.7) will be proved via Billingsley (1968) Theorem 4.2 if we show for $f \ge 0$, continuous with compact support in $\{z: |z| > a\}$ that for $\eta > 0$ and $\ell = 1, \dots, h$

$$(3.13)\qquad \lim_{s\to\infty} \limsup_{n\to\infty} P\Big[\sum_{k=1}^{n} f(a_n^{-1} Z_k Z_{k+\ell})\, 1_{\{|Z_{k+\ell}| \le s] \cup [a_n^{-1}|Z_{k+\ell}| > s^{-1}]\}^c} > \eta\Big] = 0.$$

The probability in (3.13) is bounded by

$$nP[a_n^{-1}|Z_k Z_{k+\ell}| > a,\ s < |Z_{k+\ell}| \le a_n/s]$$

and the desired result thus follows from (3.2). $\square$

We now sum the points in Proposition 3.4.

<u>Theorem 3.5</u>. Suppose (1.1), (1.2), (3.1) hold, $E|Z_1|^\alpha < \infty$, $0 < \alpha < 2$ and set $b_n = E\, Z_1 Z_2 1_{[|Z_1 Z_2| \le a_n]}$. Then in $\mathbb{R}^{h+1}$

(i) For $0 < \alpha < 1$

$$\sum_{k=1}^{n} (a_n^{-2} Z_k^2,\ a_n^{-1} Z_k Z_{k+\ell},\ \ell = 1, \dots, h)$$

$$\Rightarrow \Big(\sum_{k=1}^{\infty} j_k^2,\ \sum_{k=1}^{\infty} j_k (Z_{-\ell}^{(k)} + Z_\ell^{(k)}),\ \ell = 1, \dots, h\Big)$$

$$=: (S_0,\ \xi_{-\ell} + \xi_\ell,\ \ell = 1, \dots, h)$$

$$=: (S_0,\ S_\ell,\ \ell = 1, \dots, h)$$

(ii) For $1 \leq \alpha < 2$

$$\sum_{k=1}^{n} (a_n^{-2} Z_k^2,\ a_n^{-1}(Z_k Z_{k+\ell} - b_n),\ \ell = 1, \ldots, h)$$

$$\Rightarrow \Big(\sum_{k=1}^{\infty} j_k^2,\ \{\sum_{k=1}^{\infty} j_k Z_{-\ell}^{(k)} 1_{[|j_k Z_{-\ell}^{(k)}| > 1]}$$

$$+ \lim_{\delta \to 0} \Big(\sum_{k=1}^{\infty} j_k Z_{-\ell}^{(k)} 1_{[|j_k Z_{-\ell}^{(k)}| \in (\delta, 1]]} - \alpha E|Z_1|^{\alpha}(\delta^{-(\alpha-1)} - 1)/(\alpha - 1)\Big)\}$$

$$+ \{\sum_{k=1}^{\infty} j_k Z_{\ell}^{(k)} 1_{[|j_k Z_{\ell}^{(k)}| > 1]}$$

$$+ \lim_{\delta \to 0}\Big(\sum_{k=1}^{\infty} j_k Z_{\ell}^{(k)} 1_{[|j_k Z_{\ell}^{(k)}| \in (\delta, 1]]} - \alpha E|Z_1|^{\alpha}(\delta^{-(\alpha-1)} - 1)/(\alpha - 1)\Big)\},\ \ell = 1, \ldots, h\Big)$$

$$=: (S_o,\ \xi_{-\ell} + \xi_{\ell},\ \ell = 1, \ldots, h)$$

$$=: (S_o,\ S_{\ell},\ \ell = 1, \ldots, h).$$

(For $\alpha = 1$, interpret the above formulas in the obvious way; for instance by letting $\alpha \downarrow 1$.)

Remark: The variables $(S_o, \xi_{-\ell}, \xi_{\ell}, \ell = 1, \ldots, h)$ are dependent due to the presence of the j_k's in each. Each of the variables S_o, $\xi_{-\ell}$, ξ_{ℓ}, S_{ℓ}, $\ell = 1, \ldots, h)$ is stable; S_o is stable with index $\alpha/2$ and the rest are stable with index α. The representation of $\xi_{-\ell}$ or ξ_{ℓ} given above is the Ito representation of an infinitely divisible random variable, cf. Ito (1969).

Proof. Based on Proposition 3.4, it is clear how to proceed; cf. Davis (1983), Resnick (1984). Continuously map the points which in absolute value are bigger than δ of the point processes in Proposition 3.4 into the sum of the points. Adding a centering for the case $1 \leq \alpha < 2$ we obtain

$$\sum_{k=1}^{n} (a_n^{-2} Z_k^2 1_{[|a_n^{-2} Z_k^2| > \delta]}, \quad a_n^{-1} Z_k Z_{k+\ell} 1_{[|a_n^{-1} Z_k Z_{k+\ell}| > \delta]} -$$

$$Ea_n^{-1} Z_k Z_{k+\ell} 1_{[|a_n^{-1} Z_k Z_{k+\ell}| \varepsilon (\delta, 1]]}, \ \ell = 1, \ldots, h)$$

$$\Rightarrow \sum_{k=1}^{\infty} j_k^2 1_{[|j_k| > \delta]}, \sum_{k=1}^{\infty} (j_k Z_{-\ell}^{(k)} 1_{[|j_k Z_{-\ell}^{(k)}| > \delta]} + j_k Z_{\ell}^{(k)} 1_{[|j_k Z_{\ell}^{(k)}| > \delta]})$$

$$- 2\alpha E|Z_1|^{\alpha}(\delta^{-(\alpha-1)} - 1)/(\alpha - 1), \ \ell = 1, \ldots, h).$$

As $\delta \to 0$, the right side above converges weakly to the desired limit in (ii). The result will be proved via Billingsley's (1968) Theorem 4.2 if we prove for any $\eta > 0$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} P\Big[\Big|\sum_{k=1}^{n} a_n^{-1}(Z_k Z_{k+\ell} - b_n) - \sum_{k=1}^{n} a_n^{-1}(Z_k Z_{k+\ell} 1_{[|a_n^{-1} Z_k Z_{k+\ell}| > \delta]}$$

$$- E Z_k Z_{k+\ell} 1_{[|a_n^{-1} Z_k Z_{k+\ell}| \varepsilon (\delta, 1]]})\Big| > \eta\Big]$$

$$= \lim_{\delta\to 0} \limsup_{n\to\infty} P\Big[\Big|\sum_{k=1}^{n} a_n^{-1}(Z_k Z_{k+\ell} 1_{[|a_n^{-1} Z_k Z_{k+\ell}| \le \delta]} -$$

$$- E Z_k Z_{k+\ell} 1_{[|a_n^{-1} Z_k Z_{k+\ell}| \le \delta]})\Big| > \eta\Big] = 0.$$

Since $\{Z_k Z_{k+\ell}, -\infty < k < \infty\}$ is ℓ-dependent, this is handled as in Davis (1983), pages 265, 266.

The case $0 < \alpha < 1$ is handled similarly without the need for centering. $\square$

We now consider the weak limit behavior of the sample correlation function of moving averages of the Z_t's. Define as before $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ where $\{Z_t\}$ satisfies (1.1), (1.2), (3.1), $E|Z_1|^{\alpha} < \infty$, $0 < \alpha < 2$ and

$$\text{(3.13)} \qquad \sum_{j=-\infty}^{\infty} |c_j|^{\alpha}|j| < \infty \qquad \text{if } 0 < \alpha < 1$$

$$\sum_{j=-\infty}^{\infty} |c_j|^{1-\varepsilon}|j| < \infty \qquad \text{for some } 0 < \varepsilon < 1, \text{ if } \alpha = 1$$

$$\sum_{j=-\infty}^{\infty} |c_j|^{\alpha}\,|j| < \infty \quad \text{and} \quad \sum_{j=-\infty}^{\infty} |c_j| < \infty \quad \text{if } 1 < \alpha < 2.$$

These assumptions on the coefficients $\{c_j\}$, guarantee that $\{X_t\}$ exists as a stationary process since

$$E\left|\sum_j c_j\, Z_{t-j}\right|^{\alpha} \le \sum_j |c_j|^{\alpha} E|Z_1|^{\alpha} < \infty \qquad \text{for } 0 < \alpha \le 1$$

and

$$E\left|\sum_j c_j\, Z_{t-j}\right| \le \sum_j |c_j| E|Z_1| < \infty \qquad \text{for } 1 < \alpha < 2.$$

Recall the sample correlation function is

$$\hat{\rho}(h) = C(h)/C(0), \qquad h \ge 0$$

where

$$C(h) = \sum_{t=1}^{n} X_t\, X_{t+h}$$

and as before we set $\rho(h) = \sum_j c_j\, c_{j+h}/\sum_j c_j^2$. In Davis and Resnick (1984a) it was shown that $\hat{\rho}(h) \xrightarrow{P} \rho(h)$. Here, we shall show that $a_n(\hat{\rho}(h) - \rho(h))$ converges in distribution where $\{a_n\}$ is defined in (3.3).

The first step is to verify that

$$(3.14) \qquad a_n^{-2}\left(C(0) - \sum_{j=-\infty}^{\infty} c_j^2 \sum_{t=1}^{n} Z_t^2\right) \xrightarrow{P} 0$$

and

$$(3.15) \qquad a_n^{-2}(C(0)) \Rightarrow \sum_{j=-\infty}^{\infty} c_j^2\, S_o$$

where S_0 is defined in Theorem 3.5. If we set $\psi_{i,j} = c_i(c_{i-j+\ell} - c_{i-j}\rho(\ell))$ for $i = 0, \pm 1, \pm 2, \ldots$, $j = \pm 1, \pm 2, \ldots$ then we get for every positive integer ℓ,

$$(3.16) \qquad a_n\left(\hat{\rho}(\ell) - \rho(\ell) - (C(0))^{-1} \sum_{t=1}^{n} \sum_{j\ne 0} \sum_{i=-\infty}^{\infty} \psi_{i,j}\, Z_{t-i}\, Z_{t-i+j}\right) \xrightarrow{P} 0$$

and for each $j > 0$

$$(3.17)\qquad a_n^{-1}\Big(\sum_{t=1}^{n}\Big(\sum_{i=-\infty}^{\infty}\psi_{i,j}\,Z_{t-i}\,Z_{t-i+j}+\sum_{i=-\infty}^{\infty}\psi_{i,-j}\,Z_{t-i}\,Z_{t-i-j}\Big)$$

$$-\sum_{i=-\infty}^{\infty}(\psi_{i,j}+\psi_{i,-j})\sum_{t=1}^{n}Z_t\,Z_{t+j}\Big)\overset{P}{\to}0.$$

The proofs of (3.14) - (3.17) are practically identical to the proofs of the corresponding results given in Propositions 4.1 - 4.3 of Davis and Resnick (1984b) (by either setting $\delta = \alpha$ or $\delta = 1$), and hence are omitted. Our main result on limit distributions for sample correlation functions is now stated.

Theorem 3.6. Suppose $X_t = \sum_j c_j Z_{t-j}$ where $\{c_j\}$ satisfies (3.13) and $\{Z_t\}$ satisfies (1.1), (1.2), (3.1), $E|Z_1|^\alpha < \infty$ and $0 < \alpha < 2$. Let a_n be given by (3.3) and set $b_n = E\, Z_1 Z_2 1_{[|Z_1 Z_2| \leq a_n]}$ if $\alpha \geq 1$ and $b_n = 0$ for $\alpha < 1$. Define for $\ell > 0$, $d_{\ell,n} = 2n\sum_{j=1}^{\infty}(\rho(\ell+j)+\rho(\ell-j)-2\rho(j)\rho(\ell))(\sum_{i=-\infty}^{\infty}c_i^2)b_n$ and $Y_\ell =$

$\sum_{j=1}^{\infty}(\rho(\ell+j)+\rho(\ell-j)-2\rho(j)\rho(\ell))\,S_j/S_o$ where $(S_o, S_j, j = 1, 2, \ldots)$ is defined in Theorem 3.5. Then in $\mathbb{R}^h$,

(i) For $0 < \alpha < 1$,

$$(3.18)\qquad (a_n(\hat\rho(\ell)-\rho(\ell)), 1 \leq \ell \leq h) \Rightarrow (Y_1, Y_2, \ldots, Y_h).$$

(ii) For $1 \leq \alpha < 2$,

$$(3.19)\qquad (a_n(\hat\rho(\ell)-\rho(\ell)-d_{\ell,n}/C(0)), 1 \leq \ell \leq h) \Rightarrow (Y_1, Y_2, \ldots, Y_h).$$

If either $1 < \alpha < 2$ and $E\, Z_1 = 0$ or if $\alpha = 1$ and Z_1 is symmetric about zero then (3.19) holds with $d_{\ell,n} = 0$, $\ell = 1, \ldots, h$ and a location change in the Y_j's.

Proof. The proof follows the proof of Theorem 4.4 in Davis and Resnick (1984b). First, from Theorem 3.5 and (3.14), (3.16) and (3.17) we have for any fixed positive integer m,

$$(3.20)\qquad \Big(a_n^{-2}C(0),\; a_n^{-1}\sum_{0<|j|\leq m}\sum_{t=1}^{n}\Big(\sum_{i=-\infty}^{\infty}\psi_{i,j}(Z_{t-i}\,Z_{t-i+j}-b_n)\Big)\Big)$$

$$\Rightarrow\Big(\sum_{i=-\infty}^{\infty}c_i^2\,S_0,\;\sum_{j=1}^{m}\sum_{i=-\infty}^{\infty}(\psi_{i,j}+\psi_{i,-j})S_j\Big)$$

The next step in the proof is to show that (3.20) remains valid with m replaced by ∞ and then use (3.16) to determine the weak limit of $a_n(\hat{\rho}(h) - \rho(h))$.

The limit in (3.20) is true with m replaced by ∞ provided (cf. Theorem 4.2, Billingsley, 1968) that

$$(3.21)\qquad \lim_{m\to\infty} \limsup_{n\to\infty} P(a_n^{-1}|\sum_{|j|>m} \sum_{t=1}^{n} \sum_{i=-\infty}^{\infty} \psi_{i,j}(Z_{t-i} Z_{t-i+j} - b_n)| > \gamma) = 0$$

for every $\gamma > 0$ and

$$(3.22)\qquad \sum_{j=1}^{m} \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j})S_j \Rightarrow \sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j})S_j.$$

The verification of (3.21) is the same as in the proof of Theorem 4.4 in Davis and Resnick (1984b) and hence is omitted. As for (3.22) we consider three separate cases. For $0 < \alpha < 1$, $\sum_{j=1}^{m} \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j})S_j$ has a stable distribution with Lévy measure (cf. Resnick, 1984) given by $E\left|\sum_{j=1}^{m} \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j})Z_j\right|^{\alpha} \lambda(dx)$ where recall $\lambda(dx)$ is the mean measure of the PRM $\sum_k \varepsilon_{j_k}$. Now since $E\left|\sum_{j=1}^{\infty} (\sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j})Z_j)\right|^{\alpha} \le \sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} (|\psi_{i,j}|^{\alpha} + |\psi_{i,-j}|^{\alpha})E|Z_1|^{\alpha} < \infty$, the Lévy measures converge and hence the corresponding characteristic functions in (3.22) converge as desired. Also, for $\alpha = 1$

$$E\left|\sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j})S_j\right|^{1-\varepsilon} \le \sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} (|\psi_{i,j}|^{1-\varepsilon} + |\psi_{i,-j}|^{1-\varepsilon})E|S_1|^{1-\varepsilon} < \infty$$

and for $1 < \alpha < 2$

$$E|\sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j})S_j| \le \sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} (|\psi_{i,j}| + |\psi_{i,-j}|)E|S_1| < \infty$$

so that we have a.s. convergence in (3.22) for $1 \le \alpha < 2$.

To complete the proof, observe that $d_{\ell,n} = n\sum_{j\neq 0} \sum_{i=-\infty}^{\infty} (\psi_{i,j}+\psi_{i,-j})$, and hence

by (3.14) - (3.17) and the continuous mapping theorem

$$a_n(\hat{\rho}(\ell) - \rho(\ell) - d_{\ell,n}/C(0)) = a_n^2(C(0))^{-1}\, a_n^{-1} \sum_{t=1}^{n} \sum_{j\neq 0} \sum_{i=-\infty}^{\infty} \psi_{i,j}(Z_{t-i}\, Z_{t-i+j} - b_n) + o_p(1)$$

$$\Rightarrow \sum_{j=1}^{\infty} \sum_{i=-\infty}^{\infty} (\psi_{i,j} + \psi_{i,-j}) S_j / (\sum_{i=-\infty}^{\infty} c_i^2\, S_o)$$

$$= Y_\ell .$$

The proof of joint convergence is a straightforward extension of the above argument. □

Following Davis and Resnick (1984b) we can also derive the limit laws of the mean corrected version of the sample correlation function defined by

$$\tilde{\rho}(\ell) = \sum_{t=1}^{n} (X_t - \bar{X})(X_{t+\ell} - \bar{X}) / \sum_{t=1}^{n} (X_t - \bar{X})^2$$

where $\bar{X} = \sum_{t=1}^{n} X_t/n$. For $1 < \alpha < 2$ we have

$$(a_n(\tilde{\rho}(\ell) - \rho(\ell),\ 1 \le \ell \le h) \Rightarrow (Y_\ell,\ 1 \le \ell \le h)$$

and for $0 < \alpha < 1$

$$(n(\tilde{\rho}(\ell) - \rho(\ell)),\ 1 \le \ell \le h) \Rightarrow ((\rho(\ell) - 1),\ 1 \le \ell \le h)(\sum_{i=-\infty}^{\infty} c_i)^2 S^2 / (\sum_{i=-\infty}^{\infty} c_i^2\, S_o)$$

where $S = \sum_{k=1}^{\infty} j_k$.

<u>Remark</u>. Define $\tilde{a}_n = \inf\{x: P(|Z_1 Z_2| > x) \leq n^{-1}\}$. We now show that (3.1) is necessary in order for $\tilde{a}_n^{-1} \sum_{t=1}^{n} Z_t Z_{t+1}$ to have a limit distribution. Suppose for the sake of this argument that Z_1 has a symmetric distribution satisfying (1.1) and $E|Z_1|^\alpha < \infty$ and that $\tilde{a}_n^{-1} \sum_1^n Z_t Z_{t+1}$ converges in distribution. It follows that $Z_1 Z_2$ also has a symmetric distribution, that $|Z_1 Z_2|$ has regularly varying tail probabilities (Embrechts and Goldie, 1980) and therefore that $Z_1 Z_2$ belongs to the domain of attraction of a symmetric α-stable distribution. We first note that

(3.23) $$\frac{P(|Z_1| > t)}{P(|Z_1 Z_2|>t)} \quad \text{converges as } t \to \infty$$

if and only if

$$\frac{P(|Z_1| > \tilde{a}_n)}{P(|Z_1 Z_2|>\tilde{a}_n)} \sim n\, P(|Z_1| > \tilde{a}_n) \text{ converges as } n \to \infty.$$

From the argument given at the beginning of Section 3, we know that $\liminf_{t\to\infty} \frac{P(|Z_1 Z_2|>t)}{P(|Z_1| > t)} \geq 2\, E|Z_1|^\alpha$. Thus, suppose $n' P(|Z_1| > \tilde{a}_{n'}) \to \theta$ for some subsequence $n' \to \infty$ where $0 \leq \theta \leq (2E|Z_1|^\alpha)^{-1}$. We shall show

(3.24) $$\tilde{a}_{n'}^{-1} \sum_{t=1}^{n'} Z_t Z_{t+1} \Rightarrow S(g(\theta))^{1/\alpha}$$

where S is the symmetric α-stable random variable with characteristic function $e^{-|t|^\alpha}$ and $g \geq 0$ is a 1-1 function of θ. Once this is established, then clearly the convergence in distribution of $\sum_{t=1}^{n} Z_t Z_{t+1}/\tilde{a}_n$ will preclude the two distinct limit values in (3.23).

If $\theta = 0$ then with a_n defined as in (3.3) it is easy to show that $\tilde{a}_{n'}/a_{n'} \to \infty$. The argument in Section 3 of Davis and Resnick (1984b) can now be easily adapted to show that (3.24) is valid with $g(0) = 1$. So now suppose $\theta > 0$ and for a fixed $k > 3$ let $\underset{\sim}{W} = (Z_1 Z_2, Z_2 Z_3, \ldots, Z_{k-1} Z_k)$. We first show that for $A \in \mathcal{B}(\mathbb{R}^{k-1} - \{0\})$,

(3.25) $$n' P(\tilde{a}_{n'}^{-1} \underset{\sim}{W} \in A) \to \nu(A) \text{ for all } A \text{ with } \nu(\partial A) = 0 \text{ and } \nu(A) < \infty$$

where ν is a Lévy measure defined as follows: First define measures ν_1, ν_2, ν_3 on $\mathbb{R}^2 - \{0\}$ such that for a set $A = (x, \infty) \times (y, \infty)$ in the interior of the first quadrant

$$\nu_1(A) = \nu_2(A) = \nu_3(A) = 4^{-1}\,\theta\, E\left(\frac{|Z_1|}{x} \wedge \frac{|Z_2|}{y}\right)^{\alpha}, \quad x > 0,\ y > 0$$

and then extend the measures symmetrically to the interiors of the other three quadrants. On the axes define for $x > 0$

$$\nu_1((x, \infty) \times \{0\}) = \nu_3(\{0\} \times (x, \infty)) = \tfrac{1}{2}x^{-\alpha}(1 - \theta\, E|Z_1|^{\alpha})$$

$$\nu_1(\{0\} \times (x, \infty)) = \nu_2((x, \infty) \times \{0\})$$

$$= \nu_2(\{0\} \times (x, \infty)) = \nu_3((x, \infty) \times \{0\}) = \tfrac{1}{2}x^{-\alpha}(1 - 2\theta\, E|Z_1|^{\alpha})$$

with a symmetric definition on the negative side of the axes. Now for a set $A \in \mathcal{B}(\mathbb{R}^{k-1} - \{0\})$, $\nu(A)$ is defined to be

$$\nu(A) = \nu_1(P_1 \cap A) + \sum_{i=2}^{k-3} \nu_2(P_i \cap A) + \nu_3(P_{k-2} \cap A) - \sum_{i=2}^{k-2} \nu_2(e_i \cap A)$$

where P_i is the plane formed by the x_i and x_{i+1} axes and the intersection $P_i \cap A$ is interpreted as a two-dimensional set. The set $e_i \cap A$ is meant to represent the two dimensional set $(\mathbb{R} \times \{0\}) \cap (P_i \cap A)$, and the ν_2 measure of these sets are subtracted off since $P_i \cap P_{i+1}$ is equal to the x_{i+1} axis and hence should only contribute once in the sum.

It is easy to see that if A is not a subset of $\bigcup_{i=1}^{k-2} P_i$, then $n'P(\tilde{a}_{n'}^{-1}\, \underset{\sim}{W} \in A) \to \nu(A) = 0$ since the non-consecutive components of $\underset{\sim}{W}$ are independent. On the other hand if $A = \{\underset{\sim}{x}\colon u_1 \le x_i,\ u_2 \le x_{i+1}\}$ with $u_1 > 0$ and $u_2 > 0$, then

$$n'P(\tilde{a}_{n'}^{-1}\, \underset{\sim}{W} \in A) = n'P(Z_1 Z_2 > u_1 \tilde{a}_{n'},\ Z_2 Z_3 > u_2 \tilde{a}_{n'})$$

which by symmetry

$$= 4^{-1} n'P(|Z_1 Z_2| > u_1 \tilde{a}_{n'},\ |Z_2 Z_3| > u_2 \tilde{a}_{n'})$$

$$= 4^{-1} n'P\left(|Z_2|\left(\frac{|Z_1|}{u_1} \wedge \frac{|Z_3|}{u_2}\right) > \tilde{a}_{n'}\right)$$

$$\sim 4^{-1} E\left(\frac{|Z_1|}{u_1} \wedge \frac{|Z_3|}{u_2}\right)^{\alpha} n'P(|Z_1| > \tilde{a}_{n'})$$

$$\to \nu_1((u_1, \infty) \times (u_2, \infty)) = \nu(A)$$

where the asymptotic equivalence line follows from Breiman (1965) since $E\left(\frac{|Z_1|}{u_1} \wedge \frac{|Z_2|}{u_2}\right)^{\delta} < \infty$ for some $\delta > \alpha$. Also if $A = \{\underset{\sim}{x}: u < x_i\}$ and $u > 0$, then it is easy to check that

$$\nu(A) = \begin{cases} \nu_1((u, \infty) \times \mathbb{R}) & i = 1 \\ 2\nu_2((u, \infty) \times \mathbb{R}) - \nu_2(\{0\} \times (u, \infty)) & i = 2, \ldots, k-2 \\ \nu_3(\mathbb{R} \times (u, \infty)) & i = k-1 \end{cases}$$

$$= \tfrac{1}{2}u^{-\alpha}$$

and hence $n'P(\tilde{a}_{n'}^{-1} \underset{\sim}{W} \in A) = n'P(Z_1 Z_2 > \tilde{a}_{n'}u) \to \nu(A)$. Using symmetry and the fact that the support of ν is contained in P_i, $i = 1, \ldots, k-2$, (3.25) follows.

For each fixed integer $k > 3$, we have

$$\sum_{t=1}^{n'} Z_t Z_{t+1} = \sum_{i=1}^{r} U_i + \sum_{i=1}^{r-1} V_i + Y_r$$

where $U_i = (Z_{(i-1)k+1} Z_{(i-1)k+2} + \cdots + Z_{ik-1} Z_{ik})$, $V_i = Z_{ik} Z_{ik+1}$, $Y_r = (Z_{rk} Z_{rk+1} + \cdots + Z_{n'} Z_{n'+1})$ and $r = [n'/k]$. The U_i's are iid and since U_1 has a symmetric distribution, we have from (3.25)

$$n'P(|U_1| > \tilde{a}_{n'} z) \to \nu\left(\{\underset{\sim}{x} \in \mathbb{R}^{k-1} - \{0\}: |\sum_{i=1}^{k-1} x_i| > z\}\right)$$

$$= z^{-\alpha}\nu\left(\{\underset{\sim}{x}: |\sum_{i=1}^{k-1} x_i| > 1\}\right)$$

$$= z^{-\alpha}(\nu_1(|x+y| > 1) + \nu_3(|x+y| > 1)$$

$$+ (k-4)\nu_2(|x+y| > 1) - (k-3)\nu_2(|x| > 1, y = 0)).$$

Setting $g_k(\theta) = \nu_1(|x+y| > 1) + \nu_3(|x+y| > 1) + (k-4)\nu_2(|x+y| > 1) - (k-3)\nu_2(|x| > 1, y = 0)$, we have

$$\tilde{a}_r^{-1} \sum_{i=1}^{r} U_i \Rightarrow S(g_k(\theta))^{1/\alpha}$$

where S is the symmetric stable distribution with index α (cf. de Haan and Resnick, 1984). Since $\tilde{a}_r/\tilde{a}_{n'} \to k^{-1/\alpha}$, we then obtain

$$\tilde{a}_{n'}^{-1} \sum_{i=1}^{r} U_i \Rightarrow S(g_k(\theta)/k)^{1/\alpha}$$

and as $k \to \infty$, $g_k(\theta)/k \to \nu_2(|x+y| > 1) - \nu_2(|x| > 1,\ y = 0) =: g(\theta)$ so that $S(g_k(\theta)/k)^{1/\alpha} \Rightarrow S(g(\theta))^{1/\alpha}$. Also since the V_i's are iid and $Y_r = 0_p(1)$, it follows that

$$\lim_{k\to\infty} \limsup_{n'\to\infty} P(|\tilde{a}_{n'}^{-1}| \sum_{t=1}^{n'} Z_t Z_{t+1} - \sum_{i=1}^{r} U_i| > \varepsilon) = 0$$

and hence by Theorem 4.2 in Billingsley (1968), we obtain

$$\tilde{a}_n^{-1} \sum_{t=1}^{n'} Z_t Z_{t+1} \Rightarrow S(g(\theta))^{1/\alpha}.$$

The value of $g(\theta)$ is computed from the following lemma

<u>Lemma</u>. If for some sequence of numbers $t_j \to \infty$,

$$\frac{P(|Z_1 Z_2| > t_j)}{P(|Z_1| > t_j)} \to \theta^{-1},\ 0 < \theta \le (2E|Z_1|^\alpha)^{-1}, \text{ then}$$

$$\frac{P(|Z_2(Z_1 + Z_3)| > t_j)}{P(|Z_1| > t_j)} \to 2\theta^{-1} + E|Z_1 + Z_3|^\alpha - 2E|Z_1|^\alpha.$$

It follows from the lemma that

$$n'P(|Z_2(Z_1 + Z_3)| > \tilde{a}_{n'}) \to 2 + \theta(E|Z_1 + Z_3|^\alpha - 2E|Z_1|^\alpha)$$

whose limit using (3.25) with $k = 4$ must also be equal to $\nu(A)$, where $A = \{\underset{\sim}{x} \in \mathbb{R}^3 - \{0\}: |x_1 + x_2| > 1\}$. But $\nu(A) = \nu_1((x, y): |x+y| > 1) + \nu_3((x, y): |x| > 1) - \nu_2((x, y): |x| > 1,\ y = 0)$ which from the definition of ν_1, ν_2 and ν_3 is equal to $\nu_2(|x + y| > 1) + \theta\, E|Z_1|^\alpha + \nu_2(|x| > 1) - \nu_2(|x| > 1,\ y = 0) = \nu_2(|x + y| > 1) + \theta\, E|Z_1|^\alpha + \theta\, E|Z_1|^\alpha$, so that $\nu_2(|x + y| > 1) = 2 + \theta(E|Z_1 + Z_3|^\alpha - 4\, E|Z_1|^\alpha)$. Hence $g(\theta) = \nu_2(|x + y| > 1) - \nu_2(|x| > 1,\ y = 0) = 1 + \theta(E|Z_1 + Z_3|^\alpha - 2\, E|Z_1|^\alpha)$ which is a 1-1 positive function of θ on

$[0, (2\, E|Z_1|^\alpha)^{-1}]$ as asserted in (3.24).

<u>Proof of lemma</u>. Let $\bar{G}(t) = P(|Z_1 + Z_3| > t)$ and $\bar{F}(t) = P(|Z_1| > t)$. Then by Feller (1971), p. 278, and Cline (1983), Lemma 1.2, $\bar{G}(t)/\bar{F}(t) \to 2$. For $s > 0$, we have

$$P(|Z_2(Z_1 + Z_3)| > t) = P(|Z_2(Z_1 + Z_3)| > t,\ |Z_2| \le t/s)$$
$$+ P(|Z_2(Z_1 + Z_3)| > t,\ |Z_1 + Z_3| < s) + P(|Z_2| > t/s,\ |Z_1 + Z_3| > s)$$
$$= \int_0^{t/s} \bar{G}(t/y)F(dy) + \int_0^s \bar{F}(t/y)G(dy) + \bar{F}(t/s)\bar{G}(s).$$

Similarly,

$$P(|Z_1\ Z_2| > t) = \int_0^{t/s} \bar{F}(t/y)F(dy) + \int_0^s \bar{F}(t/y)F(dy) + \bar{F}(t/s)\bar{F}(s).$$

Thus,

$$|P(|Z_2(Z_1 + Z_3)| > t) - 2P(|Z_1\ Z_2| > t) - \bar{F}(t)(E|Z_1 + Z_3|^\alpha - 2E|Z_1|^\alpha)|/\bar{F}(t)$$
$$\le \int_0^{t/s} |\bar{G}(t/y) - 2\bar{F}(t/y)|/\bar{F}(t)F(dy) + |E|Z_1 + Z_3|^\alpha - \int_0^s \bar{F}(t/y)/\bar{F}(t)G(dy)|$$
$$+ 2|E|Z_1|^\alpha - \int_0^s \bar{F}(t/y)/\bar{F}(t)F(dy)|$$
$$+ (\bar{G}(s) + 2\bar{F}(s))\bar{F}(t/s)/\bar{F}(t).$$

On the set $0 \le y \le t/s$, $t/y > s$ so that for s sufficiently large $|2 - \bar{G}(t/y)/\bar{F}(t/y)| < \varepsilon$ for all $0 \le y \le t/s$. Hence letting $t \to \infty$ through the sequence t_j, the above inequality is bounded by

$$\lim_{t_j \to \infty} \varepsilon \frac{P(|Z_1\ Z_2| > t_j)}{P(|Z_1| > t_j)} + |E|Z_1 + Z_3|^\alpha - \int_0^s y^\alpha G(dy)| + 2|E|Z_1|^\alpha - \int_0^s y^\alpha F(dy)|$$
$$+ s^\alpha(\bar{G}(s) + 2\bar{F}(s))$$

and now letting $s \to \infty$, the bound becomes $\varepsilon\, \theta^{-1}$ since $E|Z_1|^\alpha$ is assumed finite. $\square$

4. Summary

We now summarize the rather complete results describing the limit laws for the sample correlation function. The process under consideration is $X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$, where $\{Z_t\}$ is iid with Z_1 belonging to the domain of attraction of a stable distribution with index α, $0 < \alpha \le 2$. For simplicity assume $E Z_1 = 0$ if $1 < \alpha \le 2$ and for $\alpha = 1$ assume Z_1 has a symmetric distribution.

Case 1. $\alpha = 2$. Choose α_n and β_n so that

$$n\beta_n^{-2} E(Z_1 Z_2)^2 1_{[|Z_1 Z_2| \le \beta_n]} \to 1$$

and

$$n\alpha_n^{-2} E Z_1^2 1_{[|Z_1| \le \alpha_n]} \to 1.$$

Then for $h \ge 1$, we have

$$(\alpha_n^{-2} \sum_{t=1}^{n} Z_t^2, \beta_n^{-1} \sum_{t=1}^{n} Z_t Z_{t+1}, \ldots, \beta_n^{-1} \sum_{t=1}^{n} Z_t Z_{t+h})$$

$$\Rightarrow (1, S_1, S_2, \ldots, S_h)$$

where $S_1, S_2, \ldots$ are iid $N(0, 1)$. From this result, we obtain

$$(\beta_n^{-1} \alpha_n^2(\hat{\rho}(\ell) - \rho(\ell)), 1 \le \ell \le h) \Rightarrow (Y_\ell, 1 \le \ell \le h) \tag{4.1}$$

where

$$Y_\ell = \sum_{j=1}^{\infty} (\rho(\ell+j) + \rho(\ell-j) - 2\rho(j)\rho(\ell))S_j.$$

If $\sigma^2 = \text{Var}(Z_t) < \infty$, then we may take $\alpha_n = n^{\frac{1}{2}}\sigma$ and $\beta_n = n^{\frac{1}{2}}\sigma^2$ so that $\beta_n^{-1} \alpha_n^2 = n^{\frac{1}{2}}$. In this case, (4.1) is the same as Theorem 8.4.6 of Anderson (1971) by noting that $(Y_1, \ldots, Y_h)$ has a multivariate normal distribution with covariance matrix given by Bartlett's formula

$$r_{gk} = \sum_{j=1}^{\infty} (\rho(g+j) + \rho(g-j) - 2\rho(j)\rho(g))(\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k))$$

$$= \sum_{j=-\infty}^{\infty} (\rho(g+j)\rho(k+j) + \rho(g-j)\rho(k+j) - 2\rho(j)\rho(g)\rho(k+j)$$

$$- 2\rho(j)\rho(k)\rho(g+j) + 2\rho^2(j)\rho(g)\rho(k)).$$

Case 2. $0 < \alpha < 2$ and $E|Z_1|^\alpha = \infty$. Define the sequences $\tilde{a}_n$ and a_n by $\tilde{a}_n = \inf\{x: P(|Z_1 Z_2| > x) \le n^{-1}\}$ and $a_n = \inf\{x: P(|Z_1| > x) \le n^{-1}\}$. Then for $h \ge 1$,

$$(a_n^{-2} \sum_{t=1}^{n} Z_t^2, \tilde{a}_n^{-1} \sum_{t=1}^{n} Z_t Z_{t+1}, \ldots, \tilde{a}_n^{-1} \sum_{t=1}^{n} Z_t Z_{t+h}) \Rightarrow (S_o, S_1, \ldots, S_h)$$

where $S_o, S_1, \ldots, S_h$ are independent; S_o is stable with index $\alpha/2$ and $S_1, \ldots, S_h$ are identically distributed with an α-stable distribution. It was then shown in Davis and Resnick (1984b) that

$$(\tilde{a}_n^{-1} a_n^2(\hat{\rho}(\ell) - \rho(\ell)), 1 \le \ell \le h) \Rightarrow (Y_\ell, 1 \le \ell \le h)$$

where

$$Y_\ell = \sum_{j=1}^{\infty} (\rho(\ell+j) + \rho(\ell-j) - 2\rho(j)\rho(\ell))S_j/S_o.$$

The scaling $\tilde{a}_n^{-1} a_n^2$ can be written as $n^{1/\alpha}L_1(n)$ for some slowly varying function $L_1(\cdot)$.

Case 3. $0 < \alpha < 2$ and $E|Z_1|^\alpha < \infty$. *Further assume* (3.1) *and define* a_n by $a_n = \inf\{x: P(|Z_1| > x) \le n^{-1}\}$. Then for $h \ge 1$

$$(a_n^{-2} \sum_{t=1}^{n} Z_t, a_n^{-1} \sum_{t=1}^{n} Z_t Z_{t+1}, \ldots, a_n^{-1} \sum_{t=1}^{n} Z_t Z_{t+h}) \Rightarrow (S_o, S_1, \ldots, S_h)$$

where $S_o, S_1, \ldots, S_h$ are given in the statement of Theorem 3.5. Although S_o is $\alpha/2$-stable, and $S_1, \ldots, S_h$ are α-stable, they are no longer independent as was the situation in the above cases. Nevertheless, we still have

$$(a_n(\hat{\rho}(\ell) - \rho(\ell)), 1 \le \ell \le h) \Rightarrow (Y_\ell, 1 \le \ell \le h)$$

where

$$Y_\ell = \sum_{j=1}^{\infty} (\rho(\ell+j) + \rho(\ell-j) - 2\rho(j)\rho(\ell))S_j/S_o.$$

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